AMS 261: Probability Theory (Winter 2016)

Modes of convergence for sequences of random variables

Definitions. Consider $\mathbb{R}$–valued random variables X and $\{X_n : n = 1, 2, ...\}$ defined on a common probability space $(\Omega, \mathcal{F}, P)$. The following four definitions are commonly used to study convergence for the sequence of random variables, "$X_n \to X$ as $n \to \infty$", and to obtain various limiting results for random variables and stochastic processes.

Almost sure convergence ($X_n \to^{\text{a.s.}} X$). $\{X_n : n = 1, 2, ...\}$ converges almost surely to X if

$$P\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Convergence in probability ($X_n \to^{\text{p}} X$). $\{X_n : n = 1, 2, ...\}$ converges in probability to X if, for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\}) = 0.$$

Convergence in rth mean ($X_n \to^{r-\text{mean}} X$). $\{X_n : n = 1, 2, ...\}$ converges in mean of order $r \geq 1$ (or in rth mean) to X if

$$\lim_{n\to\infty} \text{E}(|X_n - X|^r) = 0,$$

provided $\text{E}(|X_n - X|^r) < \infty$, for each n.

Convergence in distribution: ($X_n \to^{\text{d}} X$). Denote by F_{X_n} and F_X the distribution function of X_n and X, respectively. $\{X_n : n = 1, 2, ...\}$ converges in distribution to X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for all points x at which F_X is continuous.

Equivalent definitions for almost sure convergence. We have proved that each of the following are necessary and sufficient conditions for $\{X_n : n = 1, 2, ...\}$ to converge almost surely to X.

(1) For any $\epsilon > 0$, $P(\limsup_{n\to\infty}\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\}) = 0$.

(2) For any $\epsilon > 0$, $\lim_{n\to\infty} P(\cup_{j=n}^{\infty}\{\omega \in \Omega : |X_j(\omega) - X(\omega)| > \epsilon\}) = 0$.

(3) For any $\epsilon > 0$, $\lim_{n\to\infty} P(\{\omega \in \Omega : \sup_{j\geq n} |X_j(\omega) - X(\omega)| > \epsilon\}) = 0$ (that is, $\sup_{j\geq n} |X_j - X| \to^{\text{p}} 0$, as $n \to \infty$).

Comparisons between the different types of convergence. We have shown that:

- Almost sure convergence implies convergence in probability.
- Convergence in rth mean implies convergence in probability, for any $r \geq 1$.
- Convergence in probability implies convergence in distribution.

It is also immediate from the definition that convergence in rth mean implies convergence in sth mean, for $r > s \geq 1$. No other implications hold without further assumptions on $\{X_n : n = 1, 2, ...\}$ and/or on X, as can be demonstrated with counterexamples.

Example 1 ($X_n \rightarrow^{\text{p}} X$ does not imply $X_n \rightarrow^{\text{a.s.}} X$).
Let $\{X_n : n = 1, 2, ...\}$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, P)$ such that, for each n, X_n takes the value 0 with probability $1 - n^{-1}$ and the value n with probability n^{-1}. (Note that, to define such X_n, we can take $\Omega = (0, 1]$ with the Borel σ-field, the uniform distribution for P, and set $X_n(\omega) = n$ if $0 < \omega \leq n^{-1}$, and $X_n(\omega) = 0$, otherwise). Then, from the definition, we have that $\{X_n : n = 1, 2, ...\}$ converges in probability to 0. However, using the first equivalent definition of almost sure convergence, we obtain that the sequence does not converge to 0 almost surely.

Example 2 ($X_n \rightarrow^{\text{d}} X$ does not imply $X_n \rightarrow^{\text{p}} X$).
Consider two independent random variables X and Y on $(\Omega, \mathcal{F}, P)$ both taking values 0 and 1 with probability 0.5 each. Set $X_n = Y$, for $n = 1, 2, ...$, which trivially implies that $\{X_n : n = 1, 2, ...\}$ converges in distribution to X. However, $|X_n - X| = |Y - X|$ takes values 0 and 1 with probability 0.5 each, therefore $P(|X_n - X| > \epsilon) = 0.5$ for any small ϵ, and thus $\{X_n : n = 1, 2, ...\}$ does not converge in probability to X.

Example 3 ($X_n \rightarrow^{\text{a.s.}} X$ does not imply $X_n \rightarrow^{r-\text{mean}} X$).
Let $\{X_n : n = 1, 2, ...\}$ be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ such that, for each n, X_n takes the value 0 with probability $1 - n^{-2}$ and the value n with probability n^{-2}. Then, using the second equivalent definition of almost sure convergence, we obtain that $\{X_n : n = 1, 2, ...\}$ converges almost surely to 0. However, based on the definition, the sequence does not converge in mean of order 2 (and therefore it also does not converge in mean of any order greater than 2).

Example 4 ($X_n \rightarrow^{r-\text{mean}} X$ does not imply $X_n \rightarrow^{\text{a.s.}} X$).
Let $\{X_n : n = 1, 2, ...\}$ be a sequence of independent random variables on $(\Omega, \mathcal{F}, P)$ such that, for each n, X_n takes value 0 and 1 with probability $1 - n^{-1}$ and n^{-1}, respectively. Then, from the definition, $\{X_n : n = 1, 2, ...\}$ converges to 0 in mean of order r, for any $r \geq 1$. However, using the Borel-Cantelli lemma, $P(\limsup_{n\rightarrow\infty}\{\omega \in \Omega : |X_n(\omega)| > \epsilon\}) = 1$, for any $\epsilon > 0$, and therefore the sequence does not converge almost surely.