AMS 261: Probability Theory (Winter 2016)

Modes of convergence for sequences of random variables

Consider $\mathbb{R}$–valued random variables X and $\{X_n : n = 1, 2, ...\}$ defined on a common probability space $(\Omega, \mathcal{F}, P)$.

Almost sure convergence: $\{X_n : n = 1, 2, ...\}$ converges almost surely (or with probability 1) to X if $P(\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$.

Convergence in probability: $\{X_n : n = 1, 2, ...\}$ converges in probability to X if, for any $\varepsilon > 0$, $\lim_{n\to\infty} P(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) = 0$.

Convergence in mean: $\{X_n : n = 1, 2, ...\}$ converges in mean of order r (with $r \geq 1$) to X if $\lim_{n\to\infty} \mathrm{E}(|X_n - X|^r) = 0$ (provided $\mathrm{E}(|X_n - X|^r) < \infty$, for each n).

Convergence in distribution: Denote by F_{X_n} and F_X the distribution function of X_n and X, respectively. $\{X_n : n = 1, 2, ...\}$ converges in distribution to X if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$, for any x where F_X is continuous.

Random infinite series

Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of **independent** $\mathbb{R}$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. A key theoretical question for the infinite series, $\sum_{n=1}^{\infty} X_n$, corresponding to the sequence of independent random variables X_n, involves study of conditions for almost sure (a.s.) convergence of $\sum_{n=1}^{\infty} X_n$.

By definition, $\sum_{n=1}^{\infty} X_n$ converges a.s. if the sequence of random variables $\{S_n : n = 1, 2, ...\}$, where $S_n = \sum_{i=1}^{n} X_i$, converges a.s. Hence, the definition exploits the relation between series of random variables and series of reals (note that for each $\omega \in \Omega$, $\sum_{n=1}^{\infty} X_n(\omega)$ is a series of reals). In fact, it can be shown that for independent $\{X_n : n = 1, 2, ...\}$, $\sum_{n=1}^{\infty} X_n$ either converges a.s. or diverges a.s., with some key convergence results including:

Theorem 1: For a sequence $\{X_n : n = 1, 2, ...\}$ of independent $\mathbb{R}$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$, $\sum_{n=1}^{\infty} X_n$ converges a.s. to an $\mathbb{R}$–valued random variable Z if and only if $\sum_{n=1}^{\infty} X_n$ converges in probability to Z.

Theorem 2: Consider a sequence $\{X_n : n = 1, 2, ...\}$ of independent $\mathbb{R}$–valued random variables, defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that each X_n has finite variance and that $\sum_{n=1}^{\infty} \text{Var}(X_n) < \infty$. Then $\sum_{n=1}^{\infty}\{X_n - \text{E}(X_n)\}$ converges a.s.

Kolmogorov three-series theorem: Let $\{X_n : n = 1, 2, ...\}$ be an independent sequence of $\mathbb{R}$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. For each n, consider the truncated version of X_n defined by $Y_n = X_n 1_{(|X_n| \leq b)}$, where b is a positive real constant. Then $\sum_{n=1}^{\infty} X_n$ converges a.s. if and only if each of the following three series converges:

$$(1)\ \sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n| > b) \quad (2)\ \sum_{n=1}^{\infty} \text{E}(Y_n) \quad (3)\ \sum_{n=1}^{\infty} \text{Var}(Y_n)$$

Weak and strong laws of large numbers

Laws of large numbers involve convergence results for functionals of $S_n = \sum_{i=1}^{n} X_i$ (the average, $n^{-1}\sum_{i=1}^{n} X_i$, being a standard example), where the sequence of random variables $\{X_n : n = 1, 2, ...\}$ is independent. Various versions of laws of large numbers exist, but the key results include the "Weak law of large numbers" (WLLN) (yielding convergence in probability) and the "Strong law of large numbers" (SLLN) (resulting in almost sure convergence). In fact, the two standard versions for the SLLN correspond to different sets of assumptions for the sequence $\{X_n : n = 1, 2, ...\}$.

Weak law of large numbers

Consider an independent sequence of $\mathbb{R}$–valued random variables $\{X_n : n = 1, 2, ...\}$, defined on a common probability space $(\Omega, \mathcal{F}, P)$, and let $S_n = \sum_{i=1}^{n} X_i$. Assume that for each n, $\text{E}(X_n^2) < \infty$, and that $\lim_{n\to\infty} b_n^{-2} \sum_{i=1}^{n} \text{Var}(X_i) = 0$, where $\{b_n : n = 1, 2, ...\}$ is a sequence of reals. Then $b_n^{-1}(S_n - \text{E}(S_n))$ converges to 0 in probability.

Strong law of large numbers

Consider an independent sequence of $\mathbb{R}$–valued random variables $\{X_n : n = 1, 2, ...\}$, defined on a common probability space $(\Omega, \mathcal{F}, P)$, and let $S_n = \sum_{i=1}^{n} X_i$. Moreover, let $\{b_n : n = 1, 2, ...\}$ be an increasing sequence of positive reals such that $\lim_{n\to\infty} b_n = \infty$. Assume that for each n, $\text{E}(X_n) = 0$ and $\text{E}(X_n^2) < \infty$, and that $\sum_{i=1}^{\infty} b_i^{-2}\text{E}(X_i^2) < \infty$. Then $b_n^{-1} S_n$ converges to 0 almost surely.

Kolmogorov strong law of large numbers

Consider a sequence $\{X_n : n = 1, 2, ...\}$ of independent and identically distributed random variables, defined on a common probability space $(\Omega, \mathcal{F}, P)$. If $\text{E}(|X_1|) < \infty$, then $n^{-1}\sum_{i=1}^{n} X_i$ converges to $\text{E}(X_1)$ almost surely. Moreover, if $\text{E}(|X_1|) = \infty$, then $n^{-1}\sum_{i=1}^{n} X_i$ diverges a.s.