AMS 261: Probability Theory (Winter 2016)

Homework 4 (due Thursday March 3)

1. Consider a sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence is (pointwise) increasing, that is, for all n and for each $\omega \in \Omega$, $X_n(\omega) \leq X_{n+1}(\omega)$. Moreover, assume that $\mathrm{E}(X_1) > -\infty$. Denote by X the pointwise limit of $\{X_n : n = 1, 2, ...\}$, that is, for each $\omega \in \Omega$, $X(\omega) = \lim_{n\to\infty} X_n(\omega)$.
   - Prove that $\mathrm{E}(X) = \lim_{n\to\infty} \mathrm{E}(X_n)$.

2. Let $\{X_n : n = 1, 2, ...\}$ be a countable sequence of $\overline{\mathbb{R}}^+$-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$, and assume that $\mathrm{E}(\sum_{n=1}^{\infty} X_n) < \infty$.
   - Show that $\mathrm{E}\left(\sum\limits_{n=1}^{\infty} X_n\right) = \sum\limits_{n=1}^{\infty} \mathrm{E}(X_n)$.

3. Let $\{X_n : n = 1, 2, ...\}$, $\{Y_n : n = 1, 2, ...\}$, and $\{Z_n : n = 1, 2, ...\}$ be sequences of $\mathbb{R}$-valued random variables (all the random variables are defined on the same probability space). Assume that: (a) $\mathrm{E}(X_n)$ and $\mathrm{E}(Z_n)$ exist for all n and are finite; (b) each of the three sequences converges almost surely (denote by X, Y, and Z the respective almost sure limits); (c) $\mathrm{E}(X)$, $\mathrm{E}(Y)$, and $\mathrm{E}(Z)$ exist and are finite; (d) $X_n \leq Y_n \leq Z_n$ almost surely; (e) $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$, and $\lim_{n\to\infty} \mathrm{E}(Z_n) = \mathrm{E}(Z)$.
   - Show that $\lim_{n\to\infty} \mathrm{E}(Y_n) = \mathrm{E}(Y)$.

4. Let $\{X_n : n = 1, 2, ...\}$ be a countable sequence of $\mathbb{R}$–valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Assume that there exist finite real constants $p > 1$ and $K > 0$ such that $\sup_n \mathrm{E}(|X_n|^p) \leq K$.
   - Show that $\{X_n : n = 1, 2, ...\}$ is uniformly integrable.

5. Let X be an $\mathbb{R}$-valued random variable, defined on a probability space $(\Omega, \mathcal{F}, P)$, with finite expectation $\mu = \mathrm{E}(X)$ and finite standard deviation $\sigma = (\mathrm{Var}(X))^{1/2}$.
   - Prove that for any $0 \leq z \leq \sigma$,

$$P(\{\omega \in \Omega : |X(\omega) - \mu| \geq z\}) \geq \frac{\sigma^4\{1 - (z/\sigma)^2\}^2}{\mathrm{E}(|X - \mu|^4)}.$$

6. Let $\{X_n : n = 1, 2, ...\}$ be a sequence of $\mathbb{R}$-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Suppose there exists an $\mathbb{R}^+$-valued random variable Y, defined on $(\Omega, \mathcal{F}, P)$, such that $\mathrm{E}(Y) < \infty$ and $|X_n| \leq Y$, almost surely, for all n.
   - Show that $\{X_n : n = 1, 2, ...\}$ is uniformly integrable.

7. Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$-valued random variables, defined on a common probability space $(\Omega, \mathcal{F}, P)$, and an increasing function $G : [0, \infty) \to [0, \infty)$, which satisfies $\lim_{t\to\infty}\{t^{-1}G(t)\} = \infty$ and $0 < \sup_n \mathrm{E}\{G(|X_n|)\} < \infty$.
   - Prove that $\{X_n : n = 1, 2, ...\}$ is uniformly integrable.