AMS 261: Probability Theory (Winter 2016)
Convergence theorems for expectations

Monotone convergence theorem: Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}^+$–valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence is pointwise (or almost surely) increasing, that is, for all n, $X_n(\omega) \leq X_{n+1}(\omega)$ for all $\omega \in \Omega$ (or all ω in an event of probability 1). Denote by X the pointwise (or almost sure) limit of the sequence $\{X_n : n = 1, 2, ...\}$.
• Then, $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$.

Fatou lemma: Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}^+$–valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$.
• Then, $\mathrm{E}(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} \mathrm{E}(X_n)$.

Dominated convergence theorem: Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$–valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume there exists a random variable Y (also defined on $(\Omega, \mathcal{F}, P)$) such that $|X_n| \leq Y$, almost surely for all n, and $\mathrm{E}(Y) < \infty$.
• Then,

$$-\infty < \mathrm{E}(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} \mathrm{E}(X_n) \leq \limsup_{n\to\infty} \mathrm{E}(X_n) \leq \mathrm{E}(\limsup_{n\to\infty} X_n) < \infty$$

In addition to the assumptions $|X_n| \leq Y$, almost surely for all n, and $\mathrm{E}(Y) < \infty$, assume that the sequence $\{X_n : n = 1, 2, ...\}$ converges almost surely to random variable X (also defined on $(\Omega, \mathcal{F}, P)$).
• Then, $\mathrm{E}(|X|) < \infty$, $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$, and $\lim_{n\to\infty} \mathrm{E}(|X_n - X|) = 0$.

Bounded convergence theorem: Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$–valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence converges almost surely to random variable X (also defined on $(\Omega, \mathcal{F}, P)$) and that $|X_n| \leq M$, almost surely for all n, where M is a finite constant.
• Then, $\mathrm{E}(|X|) \leq M$, $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$, and $\lim_{n\to\infty} \mathrm{E}(|X_n - X|) = 0$.

Definition of uniform integrability: A sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$–valued random variables (defined on the same probability space $(\Omega, \mathcal{F}, P)$) is *uniformly integrable* if

$$\lim_{c\to\infty} \sup_n \mathrm{E}\left(|X_n| 1_{(|X_n| \geq c)}\right) = 0$$

Uniform integrability criterion: Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$–valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence converges almost surely to random variable X (also defined on $(\Omega, \mathcal{F}, P)$) and that $\mathrm{E}(|X_n|) < \infty$, for all n.
• Then the following three statements are equivalent:
(i) $\{X_n : n = 1, 2, ...\}$ is uniformly integrable
(ii) $\mathrm{E}(|X|) < \infty$ and $\lim_{n\to\infty} \mathrm{E}(|X_n - X|) = 0$
(iii) $\lim_{n\to\infty} \mathrm{E}(|X_n|) = \mathrm{E}(|X|) < \infty$
Moreover, each of (i), (ii) or (iii) implies:
(iv) $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$